STICHTING

MATHEMATISCH CENTRUM

2e BOERHAAVESTRAAT 49

AMSTERDAM

SP 103a

(S 377)

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Back to "Back to the Laplace definition"

(Statisctica Neerlandica, 23(1969), p.87-89)

Back to "Back to the Laplace definition"

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Summary The purpose of this paper is to present some corrections and supplements to the second author's papers [2] and [3] in this journal. It is a result of a discussion of the first author's criticism, which was directed against two points, viz. first the presentation of the paradoxes of Zeno from Elea ([3], § 2) and second some technical aspects of the introduction of probability by means of the notion of randomizer ([3], § 5).

The paradoxes

The main statements concerning the paradoxes the first author wants to amend are, 1. that Zeno's proofs that Achilles cannot pass the tortoise and that an arrow cannot fly are based on a mathematical model, 2. that the Greeks' faith in mathematics was so great that they never really arrived at the point of recognizing that if a model does not conform ro reality then it should not be used, 3. that many Greek philosophers thought mathematics more real than reality and that this confusion of model and reality gave rise to the Achilles paradox and to some of the others.

In fact it is commonly accepted with direct support from the sources by students of eleatic philosophy that Zeno's paradoxes are meant to show deductively that motion is only apparent and not real.

This deductive method was set forth by Parmenides, the first representative of eleatic philosophy (cf. [1], 28 B 7-8) and adopted and developed by Zeno (cf. [1], 29 B 1–4); it certainly was not meant to provide and use models in the contemporary sense.

An undeniable aspect of eleatic philosophy is a disdain for reality but there is no reason for the assumption of confusion between "model" and reality (cf. e.g. [4], [5]). Roughly speaking there are two sources for knowledge, viz. reasoning (as a source of true knowledge) and perception (as a source of unreliable knowledge) and the eleatics are in search of true knowledge by means of reasoning. Many places in the literature show that the eleatics were well aware of the distinction, cf. e.g. [1], 28 B 7–8 (Parmenides); 29 B 1–4 (Zeno); 30 B 8 (Melissos). In this respect Plato's *Parmenides* may be helpful for understanding eleatic philosophy (cf. also [4], [5]). Thus one certainly cannot maintain that the Achilles paradox arose from the confusion of model and reality. Notwithstanding Aristotle's acute criticism of the paradoxes, they should be looked upon as a fine piece of eleatic philosophy, however obscure this philosophy may be.

In defense of [3] it can be said that it may be attractive to use the paradoxes in a discussion of models and reality. In that case we should be aware that we are considering transposed paradoxes.

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The Laplace definition

The second point we wish to discuss is a question of mathematical rigour in the introduction of probability by means of a so-called randomizer together with the convention of underlining random variables (cf. [2] and [3]). It should be understood that these two aspects can be kept separate if one wishes to do so. The randomizer-Laplace definition can be used without underlining; the underlining on the other hand can be used just as well in the measure-theoretical theory of probability. Nevertheless, used in combination they support each other and therefore we introduce them together in the following recapitulation of the papers [2] and [3].

At the same time the mathematical rigour essential for an adequate presentation is improved by means of a more detailed notation.

The intuitive notion of a randomizer is amply discussed in [3], § 3 and the technical development may be sketched as follows, where we restrict ourselves to the finite case, since the general case is treated entirely similarly.

Let $\Omega = \{\omega_1, \ldots, \omega_N\}$ be a finite set. Its subsets are denoted by $A, B, \ldots$ and $N(A), N(B), \ldots$ denote the number of elements of $A, B, \ldots$. The relative frequency $f(A)$ of a subset A of Ω is defined by

$$f(A) = N(A)/N.$$

Now we consider a new symbol $\underline{\omega}$, called a randomizer on Ω, or following [2], [3] a random element on Ω.

Then we introduce expressions (possible events)

$$\underline{\omega} : \in A$$

interpreted as: the randomizer $\underline{\omega}$ takes a value in the set A*. Observe that these expressions have no truth-values, which does not matter because we are not interested in truth-values, we are interested in probabilities.

They serve as primitive expressions for our formal development. Compound expressions are defined as follows

'$\underline{\omega} : \in A$ and $\underline{\omega} : \in B$' is defined as $\underline{\omega} : \in A \cap B$

and

'$\underline{\omega} : \in A$ or $\underline{\omega} : \in B$' as $\underline{\omega} : \in A \cup B$

also

'not $\underline{\omega} : \in A$' as $\underline{\omega} : \in A'$ (the complement of A in Ω)

Now we assign probabilities to these expressions, simply by means of the Laplace definition:

$$P(\underline{\omega} : \in A) = f(A).$$

* The notation $\underline{\omega} \in A$, used in [2] and [3] is incorrect, because $\underline{\omega}$ is not an element of Ω.

The main difference with the treatment in [3], § 5 resides in the use of $:\in$ instead of $\in$, the elementhood relation of Ω, which is not defined for $\underline{\omega}$. Next consider (real) functions

$$x : \Omega \to \Pi$$

then to any such a function we associate a symbol $\underline{x}$ called a random variable. Besides we introduce expressions

$$\underline{x} : \in B,$$

where $B \subset \Pi$, with the intended practical interpretation: $\underline{x}$ takes a value in B. This has to be understood in the following way: if $\underline{\omega}$ takes the value $\omega_i \in \Omega$ then $x(\omega_i) \in B$. So when talking of a random variable $\underline{x}$ we presuppose the existence (and action) of a randomizer. From the explanation it is clear that the desired interpretation holds if we introduce $\underline{x} : \in B$ mathematically as an abbreviation of $\underline{\omega} : \in x^{-1}B$ (the inverse image of B) and this is the definition used in the mathematical model. The probability of an expression $\underline{x} : \in B$ is then, according to the definition

$$P(\underline{x} : \in B) = P(\underline{\omega} : \in x^{-1}B) = f(x^{-1}B).$$

Compound expressions involving random variables are introduced similarly to those involving $\underline{\omega}$.

If in the expression $\underline{x} : \in B$ the set B is the one-element set $\{b\}$, we may replace it by

$$\underline{x} : = b$$

If e.g. x is a real function, we can form expressions like $\underline{x} : \leqq a$, moreover real numbers may be considered as constant functions from Ω, and we can form arithmetical expressions involving random variables like $a\underline{x}+b$, etc.

With the obvious extension of the above procedure to an arbitrary probability space $(\Omega, \mathfrak{A}, P)$ we have a sound foundation of the convention of underlining random variables; the only notational difference being the use of $:\in$, $:=$, $:\leqq$, etc. instead of simply $\in$, $=$, $\leqq$. Can we take the last step and drop the ":" from our expressions? Indeed we can, because the ":" only occurs in the presence of some underlining, so we can safely omit ":" if we keep in mind the non-standard interpretation of the relation signs $\in$, $=$, $\leqq$, etc. if some underlining in the expression is present.

References

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